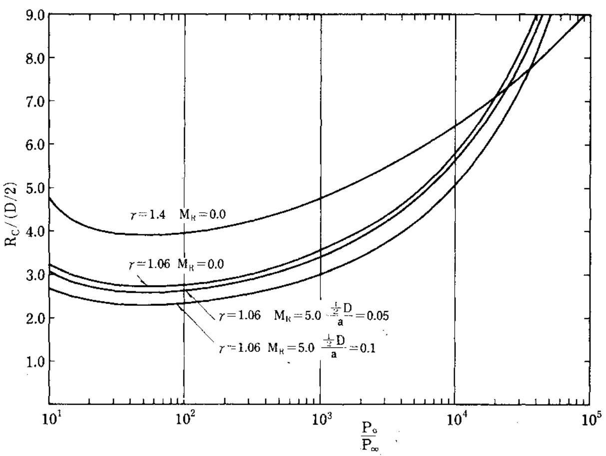


Fig. 1 Effects of expansion pressure ratio and rotational Mach number upon the initial radius of curvature of jet boundary.

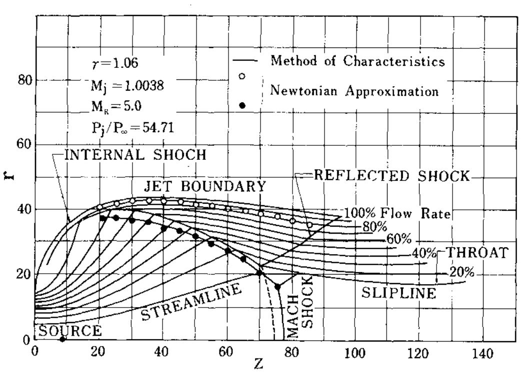
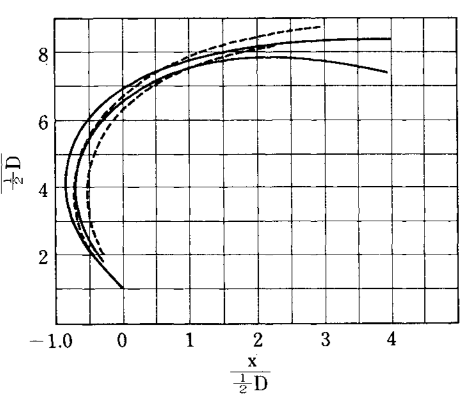


Fig. 2 Comparison of the Newtonian shock layer solution with the numerical solution by the method of characteristics. The location of Mach shock is determined by the Eastman-Radtke scheme[13] in the Newtonian approximate calculation.



$P_j/P_\infty = 632.3,\ \frac{1}{2}D/a = 0.05$

$M_R = 5,\ x_0/\frac{1}{2}D = 1.095$

Fig. 3 Comparison of the Newtonian shock layer solution with the numerical solution by the method of characteristics. ——Newtonian approximation,--- method of characteristics.

References

[1] Albini, F. A., "Approximate Computation of Underexpanded Jet Structure," *AIAA Journal*, Vol. 3, Aug. 1965, pp. 1535-1537.

[2] Hubbard, E. W., "Approximate Calculation of Highly Underexpanded Jets," *AIAA Journal*, Vol. 4, Oct. 1966, pp. 1877-1879.

[3] Boynton, F. P., "Highly Underexpanded Jet Structure: Exact and Approximate Calculations," *AIAA Journal*, Vol. 5, Sept. 1967, pp. 1703-1704.

[4] Lengrand, J., "Approximate Calculations of Rocket Plumes with Nozzle Boundary Layers and External Pressure," *Rarefied Gas Dynamics*, Vol. 1, edited by M. Becker and M. Fiebig, DFVLR-PRESS, Porz-Wahn, Germany, 1974, pp. B.13-1-B.13-10.

[5] Lengrand, J., Allegre, J., and Raffin, M., "Experimental Investigation of Underexpanded Exhaust Plumes," *AIAA Journal*, Vol. 14, May 1976, pp. 692-694.

[6] Mitome, H. and Yasuhara, M., "Simple Calculation of Highly Underexpanded Jet Plumes," *Transactions of the Japan Society for Aeronautical and Space Sciences*, Vol. 22, May 1979, pp. 44-52.

[7] Chernyi, G. G., *Introduction to Hypersonic Flow*, Academic Press, New York, 1961, pp. 136-146.

[8] Ashkenas, H. and Sherman, F. S., "The Structure and Utilization of Supersonic Free Jets in Low Density Wind Tunnels," *Rarefied Gas Dynamics*, Vol. II, edited by J. H. Leeuw, Academic Press, New York, 1964, pp. 84-105.

[9] Johannesen, N. H. and Meyer, R. E., "Axially-Symmetrical Supersonic Flow near the Centre of an Expansion," *Aeronautical Quarterly*, Vol. II, Aug. 1950, pp. 127-142.

[10] Hasegawa, H. and Mikami, H., "The Flow Near the Feed Tube Exit of a Centrifuge," Preprint of the 1978 Fall Meeting of the Atomic Energy Society of Japan, Vol. II, 1978, p. 132.

[11] Mikami, H. and Yamao, H., "Similarity between Viscons Free Jet and Source Flow Expansion with Stationary Shock Wave," *Rarefied Gas Dynamics*, Vol. II, edited by R. Campargue, CEN-Saclay, 1979, pp. 843-856.

[12] Muntz, E. P., "Measurements of Anisotropic Velocity Distribution Functions in Rapid Radial Expansions," *Rarefied Gas Dynamics*, Vol. II, edited by C. L. Brundin, Academic Press, New York, 1966, pp. 1257-1286.

[13] Eastman, D. W. and Radtke, P., "Location of the Normal Shock Wave in the Exhaust Plume of a Jet," *AIAA Journal*, Vol. 1, April 1963, pp. 918-919.

Static Stability Analysis of Elastically Restrained Structures under Follower Forces

A. N. Kounadis*
National Technical University of Athens, Athens, Greece

I. Introduction

IN this Note, elastic undamped systems carrying no attached mass and subjected to follower forces of constant magnitude are considered. These systems may be of the divergence or flutter type depending on the boundary conditions.[1] Necessary conditions for flutter and divergence instability of elastic systems under follower forces are derived in Ref. 2. Classical examples of nonconservative (in the broader sense) structural systems which can be treated as divergence type structures are Pflüger's column and Greenhill's beam.[3] In the last reference, a lower bound theorem for divergence type systems is presented using the dynamic method. According to this theorem, the critical load of a divergence type system under a follower force for certain boundary conditions is greater than the critical load of the corresponding conservative system (which is subjected to a constant directional force).

In this investigation, using as models elastically restrained simple structures under follower compressive forces, it is shown that their type of instability is dependent on the values of the constants of elastic restraint. For some values of these

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*Associate Professor, Civil Engineering Dept.

constants the structures are of the divergence type, while for others are of the flutter type. Moreover, in the case of divergence instability, it is shown that the critical load of such a structure may be smaller or greater than the critical load of the corresponding conservative system depending on the values of the constants of elastic restraint.

II. Mathematical Analysis

Consider the elastically restraint uniform beam AB of length l and flexural rigidity EI which at its end B is subjected to a follower compressive force P of constant magnitude (Fig. 1a). Let C_{RA} and C_{TA} be the stiffnesses of the rotational and translational spring at end A, respectively, and let C_{RB} and C_{TB} be the stiffnesses of the corresponding springs at end B.

Subsequently, we will try to establish the limit of stability of the column using an energy (static) approach. On the basis of linear (bifurcational) stability analysis, one may assume the existence of a nontrivial equilibrium configuration in a slightly buckled form.

Application of the principle of virtual work yields

$$\delta U+\delta\Omega+\delta\Omega^*=0 \tag{1}$$

where U is the functional of the strain energy given by

$$U=\frac{1}{2}\int_0^1 y''^2\mathrm{d}x+\frac{1}{2}\bar{C}_{TA}y^2(0)+\frac{1}{2}\bar{C}_{RA}y'^2(0)$$

$$+\frac{1}{2}\bar{C}_{TB}y^2(1)+\frac{1}{2}\bar{C}_{RB}y'^2(1) \tag{2a}$$

$$\bar{C}_{TA}=\frac{C_{TA}\ell^3}{EI},\quad \bar{C}_{RA}=\frac{C_{RA}\ell}{EI},\quad \bar{C}_{TB}=\frac{C_{TB}\ell^3}{EI},\quad \bar{C}_{RB}=\frac{C_{RB}\ell}{EI} \tag{2b}$$

Ω is the potential energy of the horizontal component of the loading which, according to the inextensionality assumption of the center line, is given by

$$\Omega=-\frac{k^2}{2}\int_0^1 y'^2\mathrm{d}x,\quad k^2=\frac{P\ell^2}{EI} \tag{3}$$

Ω^* is defined as

$$\Omega^*=k^2y'(1)y(1),\quad \delta\Omega^*=k^2y'(1)\delta y(1) \tag{4}$$

in which $-\delta\Omega^*$ is the virtual work of the vertical (nonconservative) component of the loading, and where prime denotes differentiation with respect to axial coordinate x. Integrating variational Eq. (1) by parts, and using Eqs. (2-4), the following differential equation and boundary conditions are obtained:

$$L(y)=y''''+k^2y''=0$$

$$R(y)=y''(0)-\bar{C}_{RA}y'(0)=y'''(0)+k^2y'(0)+\bar{C}_{TA}y(0)$$

$$=y''(1)+\bar{C}_{RB}y'(1)=y'''(1)-\bar{C}_{TB}y(1)=0 \tag{5}$$

where the linear differential operator L is non-selfadjoint due to the last boundary condition. For the corresponding conservative system only the last boundary condition should be changed, becoming

$$y'''(1)+k^2y'(1)-\bar{C}_{TB}y(1)=0 \tag{6}$$

Integrating Eq. (5) yields

$$y(x)=A_1\sin kx+A_2\cos kx+A_3x+A_4 \tag{7}$$

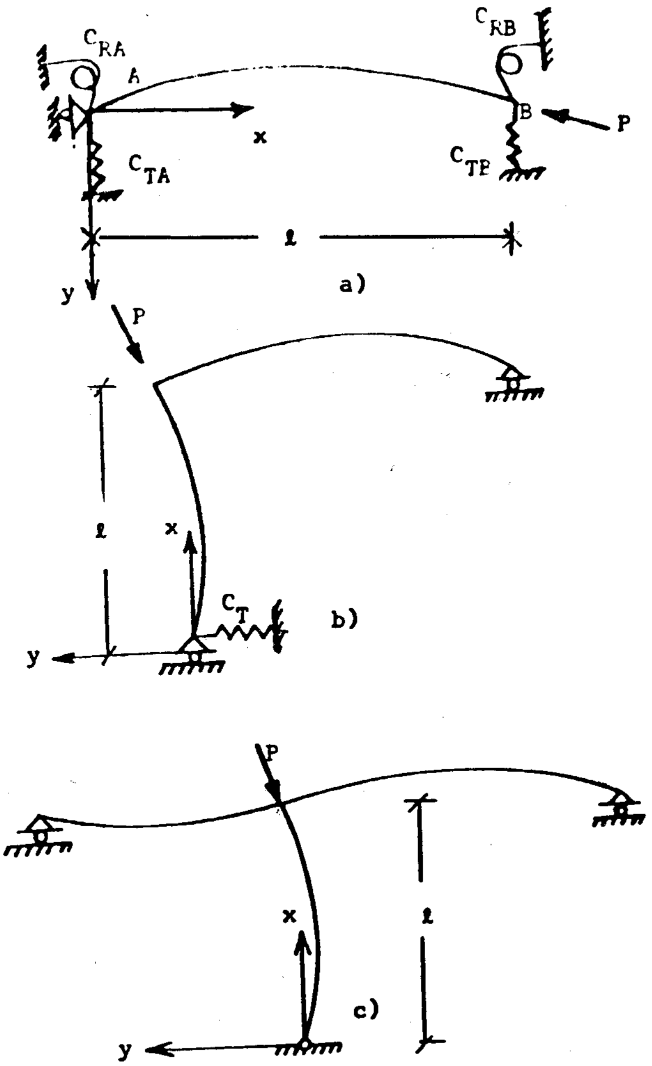

Fig. 1 a) Elastically restrained column under a follower force. b) and c) Divergence frames under a follower force.

Substituting Eq. (7) into the set of boundary conditions (5) yields a homogeneous algebraic system

$$a_{ij}A_j=0\quad i,j=1,2,3,4 \tag{8}$$

where

$$a_{11}=k\bar{C}_{RA},\ a_{12}=k^2,\ a_{13}=\bar{C}_{RA},\ a_{14}=0$$

$$a_{21}=k^2\sin k-\bar{C}_{RB}k\cos k,\ a_{22}=k^2\cos k+\bar{C}_{RB}k\sin k,$$

$$a_{23}=-\bar{C}_{RB},\ a_{24}=0.\ a_{31}=0,\ a_{32}=\bar{C}_{TA},\ a_{33}=k^2,\ a_{34}=\bar{C}_{TA}$$

$$a_{41}=\bar{C}_{TB}\sin k+k^3\cos k,\ a_{42}=\bar{C}_{TB}\cos k-k^3\sin k,$$

$$a_{43}=\bar{C}_{TB},\ a_{44}=\bar{C}_{TB} \tag{9}$$

For the corresponding conservative system only the following three terms should be changed

$$a_{41}=\bar{C}_{TB}\sin k,\ a_{42}=\bar{C}_{TB}\cos k,\ a_{43}=\bar{C}_{TB}-k^2 \tag{10}$$

For a nontrivial solution for the constants A_j, the (buckling) determinant must be zero. This leads to a transcendental equation of the form

$$F(k,\ \bar{C}_{TA},\ \bar{C}_{RA},\ \bar{C}_{TB},\ \bar{C}_{RB})=0$$

where

$$k>0,\ \bar{C}_{TA},\ \bar{C}_{RA},\ \bar{C}_{TB},\ \bar{C}_{RB}\geq 0 \tag{11}$$

When Eq. (11) holds only for $k=0$, then the column is associated with a flutter instability and the critical load should be established by using only the dynamic criterion; otherwise, the nonconservatively loaded column is associated with a divergence type instability. In this case, one should observe that flutter instability may occur before divergence (see Fig. 1 of Ref. 3); however, this does not occur in the problem considered here.

Next, some special cases are discussed which show that the effect of the stiffness constants is decisive for the type of instability of the column.

A. Conservative Systems

It is apparent that if the column is a conservative system $\Omega^*=0$ occurs either for $y'(1)=0$ or for $y(1)=0$.

The first case is obtained by setting into the buckling equation [Eq. (11)] $\bar{C}_{RB}\to\infty$, while the second is obtained by setting into this equation $\bar{C}_{TB}\to\infty$. Obviously, the values of the other stiffness constants may be arbitrary. Hence, either for $\bar{C}_{RB}\to\infty$ or $\bar{C}_{TB}\to\infty$ the column is a pure conservative system.

B. Nonconservative (Flutter Type) Systems

When the column is a pure nonconservative system there is no nontrivial equilibrium configuration, and the buckling equation [Eq. (11)] admits only the solution $k=0$. This is demonstrated through a simple example. Introducing into this equation $\bar{C}_{RB}=0$, the following buckling equation is obtained

$$\left(\frac{k^4}{\bar{C}_{RA}\bar{C}_{TA}}-\frac{k^2}{\bar{C}_{RA}}-1\right)\sin k+k\left(1-\frac{k^2}{\bar{C}_{TA}}\right)\cos k=\frac{k^3}{\bar{C}_{TB}} \tag{12}$$

For $\bar{C}_{TB}=0$, Eq. (12) admits only the trivial solution $k=0$. Hence, for $\bar{C}_{RB}=\bar{C}_{TB}=0$ the column is a pure nonconservative (flutter type) system. It corresponds to the elastically restrained Beck's column whose stability can be established only by using the dynamic criterion. In contrast, if one of the stiffness constants $\bar{C}_{RB}$ and $\bar{C}_{TB}$ is different from zero, the nonconservatively loaded column may lose its stability through divergence.

C. Divergence Systems Under Follower Forces

Excluding the previously discussed cases, there is an infinitely large number of systems for which the buckling equation admits nontrivial solutions. Some examples of them are given in Sec. III. The conclusion which can be drawn from these examples is that the divergence instability of nonconservatively loaded columns is not a rare phenomenon.

Next, a comparative study between cases A and C as far as the critical loads are concerned, is presented.

D. Critical Loads of Divergence Type Columns

In this section, a comparison is made in the carrying load capacity between divergence type columns under follower forces and the corresponding conservative columns under unidirectional forces. Here, the sign of Ω^* is examined.

For the sake of simplicity, a column with $\bar{C}_{RA}=0$ and $\bar{C}_{TA}\to\infty$ is investigated. For the column under a follower force by using Eq. (11), the following buckling equation is obtained

$$\tan k=\left[k\left(1-\frac{k^2}{\bar{C}_{TB}}\right)\right]\Big/\left[1+\frac{k^2}{\bar{C}_{RB}}\right] \tag{13}$$

In this case, Ω^* is given by

$$\Omega^*=-(A_1^2k^7/2\bar{C}_{RB}\bar{C}_{TB})\sin 2k \tag{14}$$

where A_1 is an arbitrary constant different from zero. Obviously, the sign of Ω^* is governed by the sign of $\sin 2k$.

For the corresponding conservative column with the same boundary conditions, as stated previously, the buckling equation is

$$\tan k=\left[k\left(1-\frac{k^2}{\bar{C}_{TB}}\right)\right]\Big/\left[1+\frac{k^2}{\bar{C}_{RB}}\left(1-\frac{k^2}{\bar{C}_{TB}}\right)\right] \tag{15}$$

It should be noticed that Eq. (13), independently of the values of the stiffness constants $\bar{C}_{RB}$, and $\bar{C}_{TB}$, has no root for $k<\pi/2$, and therefore, the critical (smallest) load $k_{cr}>\pi/2$. This can be easily proven by using the expansion formula

$$\tan k=k+(k^3/3)+(2/15)k^5+\ldots\text{where } k<\pi/2 \tag{16}$$

Note also that $\Omega^*\ (k_{cr})\geq 0$ implies $\tan k_{cr}\leq 0$; thus, if $\pi/2<k_{cr}<\pi$, then $\bar{C}_{TB}<\pi^2$. Denoting by $\hat{k}_{cr}$ the critical load derived from Eq. (15), then for $\pi/2\leq\bar{k}_{cr}\leq\pi/2$ the aforementioned inequalities also hold, i.e., $\tan\bar{k}_{cr}\leq 0$ and $\bar{C}_{TB}\geq\pi^2$. It can be readily shown that $|\tan\bar{k}_{cr}|\geq|\tan k_{cr}|$ which yields

$$k_{cr}\geq\bar{k}_{cr} \tag{17}$$

because in the range $[\pi/2,\pi]$ the absolute value of the tangent is a monotonically decreasing function.

From the foregoing case ($\pi/2\leq k_{cr},\bar{k}_{cr}\leq\pi$) it is observed that: 1) if $\Omega^*(k_{cr})\geq 0$, then the carrying load capacity of the column under a follower force is greater than that of the corresponding conservative column, and 2) if $\bar{C}_{TB}\geq\pi^2$, then $\tan k_{cr}>0$ and $\Omega^*(k_{cr})\geq 0$ which means that the inequality cited in Eq. (17) does not hold. Consequently, the sign of $\Omega^*(k_{cr})$ depends on the value of the stiffness constant $\bar{C}_{TB}$.

In closing this section, two more nonconservatively loaded systems, the simple frames shown in Figs. 1b and 1c, which are of the divergence type, are considered. The equation governing the lateral displacement of the column of both frames is given by

$$y(x)=B_1\sin kx+B_2\cos kx+B_3x+B_4 \tag{18}$$

From the boundary condition $y''(0)=0$ which implies $B_2=0$, Eq. (18) becomes

$$y(x)=B_1\sin kx+B_3x+B_4 \tag{19}$$

Equilibrium of forces at the joint in the horizontal direction results in

$$-EIy'''(1)-k^2y'(1)+k^2y'(1)=0 \tag{20}$$

or

$$\cos k=0 \tag{21}$$

Consequently, the critical (smallest eigenvalue) load is

$$k_{cr}^2=\pi^2/4=2.467\ldots \tag{22}$$

This value coincides with the nonlinear, divergence critical load given in Refs. 4 and 5 of the first frame for $\bar{C}_T\to\infty$ which as shown loses its stability through a bifurcation point; it coincides also with the critical load[6] obtained on the basis of the dynamic criterion.

III. Numerical Results

By solving Eq. (11) numerically, it is observed that the type of instability of a column under a follower force depends on the values of the stiffness constants $\bar{C}_{RA}$, $\bar{C}_{TA}$, $\bar{C}_{RB}$, $\bar{C}_{TB}$. Moreover, it is noticed that there is an infinite set of columns of both types of instability. This can be shown easily in the

Table 1 Divergence loads for $\bar{C}_{RB}=1$, $\bar{C}_{RA}=0$, $\bar{C}_{TA}=\infty$

No.	$\bar{C}_{TB}$	Critical load Nonconservative	Critical load Conservative	Sign of Ω^*
1	0.5	1.93	1.11	+
2	1	2.18	1.31	+
3	2	2.51	1.64	+
4	4	2.84	2.15	+
5	6	2.99	2.56	+
6	8	3.08	2.89	+
7	π^2	π	π	0
8	10	$3.142...>\pi$	3.18	−

following simple example. Consider a column for which $\bar{C}_{TB}=0$ and $\bar{C}_{RA}$, $\bar{C}_{TA}$, and $\bar{C}_{RB}$ are arbitrary constants different from zero and infinity. Corresponding to this case, the buckling equation resulting from Eq. (11) is the following:

$$\sin k+\frac{k}{\bar{C}_{RA}}\cos k+\frac{k}{\bar{C}_{RB}}=0 \tag{23}$$

This equation for $\bar{C}_{RB}=1$ and $\bar{C}_{RA}\rightarrow\infty$ admits only the trivial solution $k=0$. Consequently, this column is of the flutter type, and its critical load can be determined by using only the dynamic method. Similarly, for $\bar{C}_{RB}=1$ and $\bar{C}_{RA}>1.05$ the column is associated with a flutter type instability. In contradiction to the foregoing cases for $\bar{C}_{RA}=\bar{C}_{RB}=1$, the critical (smallest) load k_{cr} is equal to π; namely, the column is a divergence type system. Apparently, there is an infinite number of columns associated with the latter type of instability for $\bar{C}_{RB}=1$ and $\bar{C}_{RA}<1.05$.

Next, numerical results are presented in Table 1 giving a comparison between the divergence critical loads (Case D.) of a column under a follower force, and the respective conservative column. Thus, the critical loads k_{cr} and $\bar{k}_{cr}$, the sign of $\Omega^*(k_{cr})$ for $\bar{C}_{RA}=0$, $\bar{C}_{TA}\rightarrow\infty$, and $\bar{C}_{RB}=1$, and various values of $\bar{C}_{TB}$ are given. It is clear that the column under a follower force can carry a greater load than that of the corresponding conservative system only if $\Omega^*(k_{cr})\geq 0$; otherwise, $k_{cr}<\bar{k}_{cr}$. It is also shown that the sign of $\Omega^*(k_{cr})$ depends on the stiffness constants.

A particular case, where the instability mechanism of a fixed-elastically supported column under a follower load may change from flutter to divergence, and vice versa depending on the value of the stiffness constant, is presented by Sundararajan.[7] Another work of this author pertinent to this paper is given in Ref. 8. Finally, it should be noted that the lower bound theorem of Ref. 3 does not apply here.

References

[1] Herrman, G. and Bungay, R. W., "On the Stability of Elastic Systems Subjected to Nonconservative Forces," *Journal of Applied Mechanics*, Vol. 86, 1964, pp. 435-440.

[2] Plaut, R. H., "Determining the Nature of Instability in Nonconservative Problems," *AIAA Journal*, Vol. 10, July 1972, pp. 967-968.

[3] Leipholz, E. H., "Aspects of Dynamic Stability of Structures," *Journal of Engineering Mechanics Div.*, EM2, Vol. 101, 1975, pp. 109-124.

[4] Kounadis, A. N., Giri, J., and Simitses, J. G., "Divergence Buckling of a Simple Frame, Subjected to a Follower Force," *Journal of Applied Mechanics, Transactions of ASME*, 1978, Vol. 45, No. 2, pp. 426-428.

[5] Kounadis, A. N., "The Effects of Some Parameters on the Nonlinear Divergence Buckling of a Nonconservative Simple Frame," *Journal de Mécanique Appliquée*, Vol. III, No. 2, 1979, pp. 173-185.

[6] Panayotounakos, E. D. Kounadis, N. A., "Elastic Stability of a Simple Frame Subjected to a Circulatory Load," *Journal of Sound and Vibration*, Vol. 64, No. 2, 1979, pp. 179-186.

[7] Sundararajan, C., "Influence of an Elastic End Support on the Vibration and Stability of Beck's Column," *International Journal of Mechanical Sciences*, Vol. 19, 1976, pp. 239-241.

[8] Sundararajan, C., "On the Stability of Divergence Type Elastic Systems," *Developments in Theoretical and Applied Mechanics*, Vol. 7, 1974, pp. 475-482.

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Thermal Coupling of 2.8-μm Laser Radiation to Metal Targets

D. B. Nichols* and R. B. Hall†
The Boeing Aerospace Company, Seattle, Wash.

Introduction

THE response of metal targets to high-intensity laser radiation includes effects of target heating,[1] imparted momentum,[2] mass removal,[3] and the ignition of surface plasmas[4] with ionization of both atmospheric and target species;[5] all other target effects, however, are conditioned by the thermal coupling. In spite of a broad current interest in chemical lasers and their application, little is known experimentally about the thermal coupling of pulsed HF laser radiation to metals. Reported herein are coupling measurements throughout the incident fluence range 25-1400 J/cm^2.

An axially symmetric laser pulse incident on a planar target, described by a time-dependent irradiance $q_i(r,t)$ W/cm^2, results in an absorbed flux at the surface $q_a(r,t)$ W/cm^2. Integration over the time of the event gives incident fluence $e_i(r)=\int q_i(r,t)\,dt$ and absorbed fluence $e_a(r)=\int q_a(r,t)\,dt$ in J/cm^2. The total incident laser pulse energy in joules is $E_i=2\pi\int e_i(r)\,r dr$ and the total thermal energy deposited in the target $E_a=2\pi\int e_a(r)\,r\,dr$. The total thermal coupling coefficient is the fraction of incident laser pulse energy which is converted into thermal energy in the target, $\alpha=E_a/E_i$; this coefficient was measured in the present work. Time-integrated spatial dependence of coupling is expressed as local coupling $\alpha_r(r)=e_a(r)/e_i(r)$.

Laser pulses exceeding certain threshold irradiance and fluence requirements[4] ignite a surface plasma which enhances the thermal coupling to a metal target.[6] Model calculations as well as experiments have shown[7,8] the dependence of thermal coupling processes on the normalized pulse length $\hat{\tau}$. $\hat{\tau}$ is the ratio of the laser pulse length t_p to the time required for an acoustic disturbance of velocity c to cross the beam radius r_b.

$$\hat{\tau}=ct_p/r_b$$

When $\hat{\tau}$ is small (≈ 1) the local coupling coefficient $\alpha_r(r)$ is optimized because during the pulse the spatial development of the laser-supported absorption wave is limited to a region near the surface; for these conditions the energy transfer can be described by one-dimensional models.

This Note reports the utilization of the first HF-DF chemical laser delivering several hundred joules in a few microseconds, to investigate 2.8-μm thermal coupling. A first experimental requirement for spot size is that the surface

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*Research Scientist.

†Principal Scientist.